Solutions to the WIKR-06 exam of 20 June 2019

June 19, 2019

1a

$$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} \cdot \binom{x}{y} p^y (1-p)^{x-y} & x, y \text{ integers with } 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

1b

For $y \in \{0\} \cup \mathbb{N}$:

$$\begin{aligned} \mathbb{P}(Y=y) &= \sum_x \mathbb{P}(X=x, Y=y) = \sum_{x \ge y} \frac{\mu^x e^{-\mu}}{x!} \cdot \binom{x}{y} p^y (1-p)^{x-y} = \frac{(\mu p)^y}{y!} e^{-\mu} \sum_{x \ge y} \frac{(\mu(1-p))^{x-y}}{(x-y)!} \\ &= \frac{(\mu p)^y}{y!} e^{-\mu} \sum_{\ell=0}^{\infty} \frac{(\mu(1-p))^\ell}{\ell!} = \frac{(\mu p)^y}{y!} e^{-\mu} e^{+\mu(1-p)} = \frac{(\mu p)^y}{y!} e^{-\mu p}. \end{aligned}$$

We recognize this as the pmf of the $\text{Poi}(\mu p)$.

1c

We first need to find $\mathbb{E}XY$.

$$\begin{aligned} \mathbb{E}XY &= \sum_{x,y} xy \mathbb{P}(X=x, Y=y) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} xy \frac{\mu^x e^{-\mu}}{x!} \binom{x}{y} p^y (1-p)^{x-y} \\ &= \sum_{x=0}^{\infty} x \frac{\mu^x e^{-\mu}}{x!} \sum_{y=0}^{x} y \binom{x}{y} p^y (1-p)^{x-y} = \sum_{x=0}^{\infty} x^2 p \frac{\mu^x e^{-\mu}}{x!} \\ &= p \mathbb{E}X^2, \end{aligned}$$

recognizing the sum over y as the expectation of a $\text{Bin}(x,p)$. Since X is Poisson, we know from lectures (it would also be fine if you worked this out in the exam of course) that $\mathbb{E}X = \mu^2 + \mu$. Hence

$$\mathbb{E}XY = p(\mu^2 + \mu).$$

On the other hand $\mathbb{E}X = \mu, \mathbb{E}Y = \mu p, \operatorname{Var} X = \mu, \operatorname{Var} Y = \mu p$ (using 1b and our knowlegde of the Poisson distribution from lectures – or of course working out the expectation and variance of a Poisson from scratch). So

$$\rho_{X,Y} = \frac{\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y}{\sqrt{\operatorname{Var} X \operatorname{Var} Y}} = \frac{\mu^2 p + \mu p - \mu^2 p}{\sqrt{\mu^2 p}} = \sqrt{p}.$$

2a

(This pdf was used as an example in lectures – so this ought to have been an easy question for everyone.)

X, Y are **not** independent. For instance $\mathbb{P}(X > 1), \mathbb{P}(Y < 1)$ are both clearly positive, while $\mathbb{P}(X > 1, Y < 1) = 0 \neq \mathbb{P}(X > 1)\mathbb{P}(Y < 1)$.

2b

We must have

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\ dx\ dy = c \int_0^{\infty} \int_x^{\infty} e^{-y}\ dy\ dx = c \int_0^{\infty} e^{-x}\ dx = c.$$

2c

For $x > 0$ we have:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\ dy = \int_x^{\infty} e^{-y}\ dy = e^{-x}.$$

(If $x \leq 0$ then $f_X(x) = 0$.)

2d

For $y \leq 0$, $f_Y(y) = 0$ clearly. For $y > 0$:

$$f_Y(y) = \int_0^y e^{-y}\ dx = ye^{-y}.$$

2e

We set $U = X, Y = X + Y$ and use the change of variables theorem. Note

$$\begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

has determinant one. Hence

$$f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v)) = \begin{cases} e^{-(v-u)} & \text{if } 0 < u < v - u, \\ 0 & \text{otherwise.} \end{cases}$$

Now we get f_V via

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\ du = \int_0^{v/2} e^{-v+u}\ du = e^{-v/2} - e^{-v}.$$

(for $0 < v < \infty$.)

3a

All permutations of the $a+b$ votes are equally likely. The number of ways to get a vote for A last is $a \cdot (a+b-1)!$. (Choose the last vote to be one of the a votes for A and arrange the other $a+b-1$ votes in a sequence arbitrarily.) Hence

$$\mathbb{P}(\text{last ballot is vote for } A) = \frac{a \cdot (a+b-1)!}{(a+b)!} = \frac{a}{a+b}.$$

3b

Base case. The statement is clearly true when $a, b \leq 1$.
Induction step. Let a, b be arbitrary nonnegative integers, and assume the statement is true for all a', b' with $a' \leq a, b' \leq b, (a,b) \neq (a',b')$. If $a = 0$ or $b = 0$ then clearly the formula holds, so we may assume $a, b \geq 1$. Also, if $a \leq b$ then it is clearly impossible for A to be always ahead. So then the probability is zero and hence the formula is correct. We can assume $a > b \geq 1$ from now on.

We condition on the *last* vote. Let $E = \{\text{last vote is for } A\}$. Note that the probability that A was ahead at all times *given that E occurred* is the same as the probability that A is always ahead in a situation with $a-1$ votes for A and b votes for B. Hence by the induction hypothesis

$$\mathbb{P}(A \text{ always ahead} | E) = \frac{a-b-1}{a+b-1}.$$

Similarly, the probability that A was ahead at all times *given that E did not occur* is the same as the probability that A is always ahead in the situation with a votes for A and $b-1$ votes for B. So, again using the induction hypothesis:

$$\mathbb{P}(A \text{ always ahead} | E^c) = \frac{a-b+1}{a+b-1}.$$

Using 3a, it follows that

$$\begin{aligned}
\mathbb{P}(A \text{ always ahead}) &= \mathbb{P}(E)\mathbb{P}(A \text{ always ahead}|E) + \mathbb{P}(E^c)\mathbb{P}(A \text{ always ahead}|E^c) \\
&= \tfrac{a}{a+b} \cdot \tfrac{a-b-1}{a+b-1} + \tfrac{b}{a+b} \cdot \tfrac{a-b+1}{a+b-1} = \tfrac{a^2-ab-a+ab-b^2+b}{(a+b)(a+b-1)} \\
&= \tfrac{(a-b)(a+b-1)}{(a+b)(a+b-1)} = \tfrac{a-b}{a+b}.
\end{aligned}$$